Analysis of Linear Multistep Methods

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***Abstract*—Using Linear Multistep methods such as Adam-Bashforth 1 (Forward Euler Method) and Adam-Bashforth 4 we examine the behavior of first-order linear differential equations and the rigidity of our methods.**

I. Introduction

LINEAR Multistep methods are used for the numerical solution of ordinary differential equations (We will refer to these as ODEs), where an initial value is taken and successive short steps forward in time find the next solutions, creating an approximation to our ODE.

Single step methods such as Adam-Bashforth 1 (We will call this AB1, AB2 etc) are the ones we will focus on in this project. The numbers in front of the AB, such as AB1 denote the number of steps, so AB1 is a single-step method (which happens to be the Forward Euler Method). Single step methods only use one previous point and its derivative to estimate our next point. Methods such as Runge-Kutta take more points to produce a higher-order method and then subsequently discard all previous information before taking a second step.

Multistep methods try to be more accurate and efficient by not discarding the previous information and instead using it. Linear multistep methods are a linear combination of previous values and derivatives.

II. General Form

Numerical methods for ODEs attempt to approximate solutions to problems of the form;

$$y' = f(t, y), \qquad y(t_0) = y_0 \tag{1}$$

for the real-valued function y of the real variable x, where $y' = dy/dx$. and $y(t_0)$ is our initial value. The differential equation along with initial value (1) above is called an Initial Value Problem (IVP).

In general we cannot assume that our function will have a unique solution, however any elementary course on Differential Equations covers existence and uniqueness, and as such proofs will be omitted in this text.

Theorem 1: **Picard's Theorem** Suppose that f(.,.) is a continuous function of its arguments in a region U of the (x, y) plane which contains the rectangle

$$R = (x, y) : x_0 \leq x \geq X_M, \qquad |y - y_0| \leq Y_M,$$

where $X_M > x_0$ and $Y_M > 0$ are constants. Suppose also, that there exists a positive constant L such that

$$|f(x, y) - f(x, z)| \leq L|y - z| \tag{2}$$

holds whenever (x,y) and (x,z) lie in rectangle R. Finally, letting

$$M = max|f(x, y) : (x, y) \in R$$

suppose that $M(X_M - x_0) \leq Y_M$. Then there exists a unique continuously differential function $x \to y(x)$, defined on the closed interval $[x_0, X_M]$, which satisfies (1).

The condition (2) is called a **Lipschitz condition**, and L is called a **Lipshitz constant** for f.

We will quickly define some notions:
- Convergence: A numerical method is convergent if the numerical solution approaches the exact solution as the step size h goes to 0. We require that for every ODE (1) with a Lipschitz function f and every $t > 0$,

$$\lim_{h \to 0} \max_{+n=0,1,\ldots,[t^*/h]} ||y_{n,h} - y(t_n)|| = 0$$

- Consistency and Order: Suppose the Numerical Method is

$$y_{n+k} = \Psi(t_{n+k}; y(t_n), y(t_{n+1}), ..., y(t_{n+k-1}); h) - y(t_{n+k})$$

The method is said to be consistent if

$$\lim_{h \to 0} \frac{\delta^h_{n+k}}{h} = 0$$

The method is said to be of order p if

$$\delta^h_{n+k} = O(h^{p+1}) \quad \text{as} \quad h \to 0$$

Hence a method is consistent if it has an order greater than 0. The forward Euler Method has an order 1, so it is consistent. Consistency is necessary for convergence, but for a method to be convergent it also has to be zero-stable.

For some differential equations, applying methods such as Euler's method, explicit Runge-Kutta methods, Adam-Bashforth methods we may get instability in our solutions. This type of behavior is called stiffness, and is often caused by large differences in time-scales. For example a function that rapidly explodes or changes has to be taken at an appropriately small time-scale for the method to converge.

III. One-step Method

Consider

$$y' = f(t, y) = -(5p)y + t, \quad 0 \leq t \leq 1, \quad y(0) = 1$$

Where p is a given constant We can find the exact solution through direct computation, it is simple but requires a by parts evaluation on the right side. The exact value is then

$$y = \frac{t}{4p} - \frac{1}{16p^2} + \frac{1}{e^{4pt}} + \frac{1}{16p^2 e^{4pt}}$$

A. One-step Euler

A simple numerical method is Euler's method:

$$y_{n+1} = y_n + hf(t_n, y_n)$$

Euler's method can be viewed as an explicit multistep method for the degenerate case of one step.

The method, applied with a variable step sizes. We use N = 8,15,32,64,128 where $h = \frac{0 \leq t \leq 1}{N}$. We estimate the range of convergence by calculating

$$\tilde{p}_n = \frac{1}{ln(2)} ln\Big(\frac{||e_{N/2}||_\infty}{||e_N||_\infty}\Big)$$

Table I
EULER'S METHOD

N-Values	MaxError	Convergence Rate
16	2.49E-02	
32	1.24E-02	1.01E+00
64	6.16E-03	1.01E+00
128	3.07E-03	1.01E+00

Here we have a table of error values at different N values along with a plot

From this we can clearly see that Euler's method is successful at approximating this ODE and that we have a convergence rate of 1, which we showed earlier.

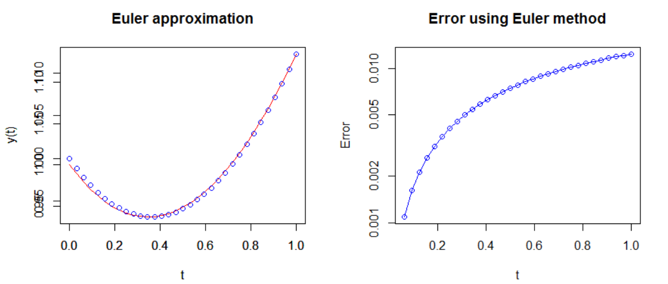

Figure 1. Euler's Method using N=32 along with Error using Euler's Method.

B. *Stiffness with Euler's Method*

We have so far evaluated the accuracy of numerical methods in terms of the rate at which our error approaches zero, when our N approaches infinity (Step-size h approaches zero). However this ignores the fact that the local truncation error of one-step and multi-step methods also depends on higher-order derivatives of the solution. In some cases, these derivatives can be very large, even when the solution is relatively small, which requires a minimum N value to be chosen to achieve accuracy.

This leads to *stiff* equations. A differential equation $y' = f(t,y)$ is said to be *stiff* if its exact solution $y(t)$ includes a term that decays exponentially to zero as t increases, but whose derivatives are much greater in magnitude than the term itself. For example e^{-ct}, where c is a large positive constant, because its nth derivative is $c^n e^{-ct}$. Because the error includes this term, the error can be quite large if our step-size is not chosen to be small enough to counter-act the large derivative (N size being sufficiently large).

Consider the problem

$$y' = -15y, y(0) = 1, 0 \leq t \leq 3$$

The exact solution this time is e^{-15t}, which rapidly decays to zero as t increases. Solving this problem using Euler's method, with step size h=0.2, we get

$$y_{n+1} = y_n - 15hy_n = -2y_n$$

which gives an exponentially growing solution $y_n = (-2)^n$. But if we chose let's say h=6.667E-3 we would obtain $y_n = (0.9)^n$ which more accurately behaves like our exact solution by rapidly decaying to zero.

We can analyze the behavior of our methods on stiff equations by applying a test equation $y' = \lambda y$. The solution is always $y(t) = e^{\lambda t}$. Since our real solution rapidly approaches zero, if our numerical method does the same for a fixed step-size our method is said to be **A-stable**.

As λ increases in value, our equation becomes increasingly stiff. We can determine how small our value of h (or N) must be for any given value of λ.
When applying a one-step method to the test equation, our computed solution is of the form

$$y_{n+1} = Q(h\lambda)y_n$$

where $Q(h\lambda)$ is a polynomial in $h\lambda$. This polynomial is supposed to approximate $e^{h\lambda}$ since our exact solution satisfies $y(t_{n+1}) = e^{h\lambda}y(t_n)$. Where we must choose h so that $|Q(h\lambda)| < 1$.

Euler's method applied to the test equation $y' = \lambda y$ is

$$y_{n+1} = y_n + hf(t_n, y_n) = y_n + h(\lambda y_n) = (1 + h\lambda)y_n$$

Hence $y_n = (1 + h\lambda)^n y_0$ with $Q(z) = 1 + z$. The region of absolute stability is then $\{Z \in \mathbb{C} | 1 = z| < 1\}$ as shown in the graph below.

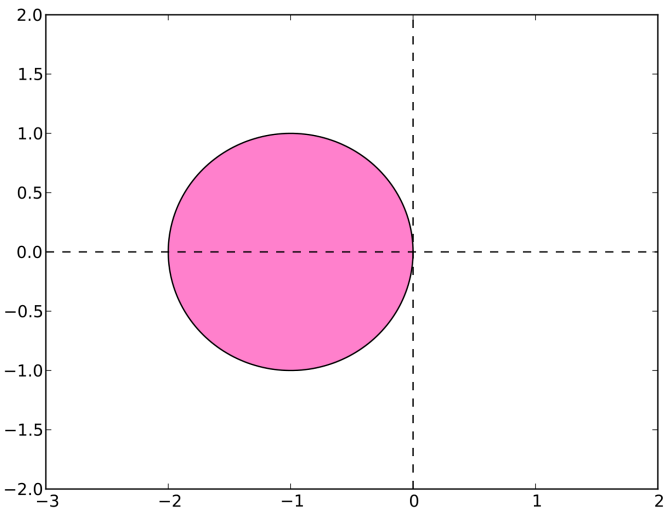

Figure 2. Circle radius 1 center (-1,0) in complex plane

The Euler method is not A-stable.

Our example had $0 \leq t \leq 3$. So what is the **minimum N** value we must take in order to get a stable solution? We have to get $-2 < z < 0$ in order to be inside our region of stability. This means, at $z = 15 * h = 2$, which is just $h = 2/15$ or $N = 22.5$ but since we can't have non-integer n, we need $N > 23$ to converge. *'The table of errors and convergence is shown on the next page.'*

Table II
EULER'S METHOD

N-Values	MaxError	Convergence Rate
8	2.09E+05	
16	1.36E+04	3.95
32	6.51E-01	1.43
64	1.98E-01	1.72
128	7.57E-02	1.39
256	3.49E-02	1.12
512	1.68E-02	1.05
1024	8.23E-03	1.03

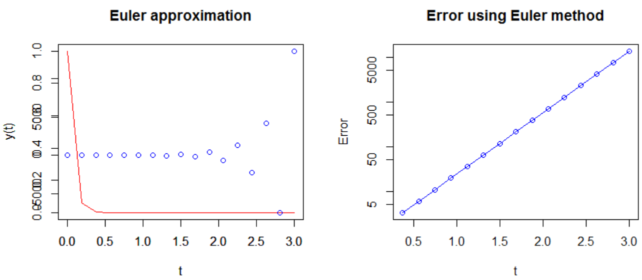

Figure 3. At N=16 we can see that our value oscillates violently

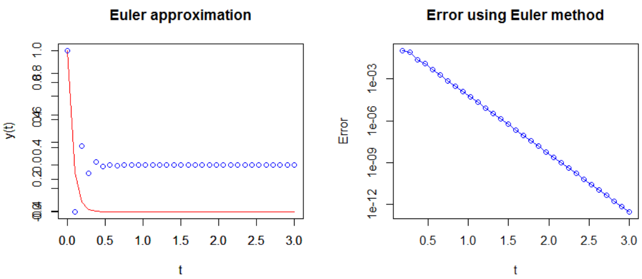

Figure 4. Here at N=32 we can see that at low t we still have oscillation

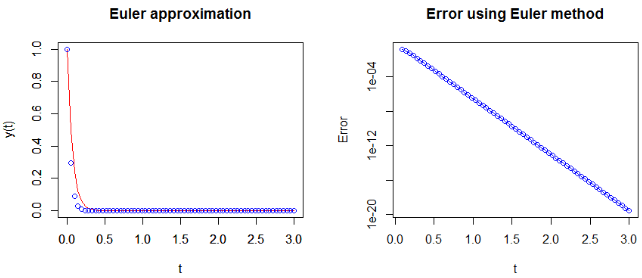

Figure 5. At N=64 Euler's method finally manages to converge at low t

IV. MULTI-STEP METHODS

The most common multi-step methods can be split into three families, Adams–Bashforth methods, Adams–Moulton methods, and the backward differentiation formulas.

We will focus on the Adams-Bashforth methods in this section. In the previous section we talked about Euler's Method, which is just the first-order Adams-Bashforth method.

These methods are used to produce predictor-corrector algorithms in which the error is controlled by varying step-size and order. To derive these methods, we use

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} f(t, y(t))dt$$

Polynomials that interpolate $y'(t) = f(t, y(t))$ are constructed and integrated over $[x_n, x_{n+1}]$ to obtain an approximation to y_{n+1}. We will focus on explicit or predictor formulas.

AB methods are based on the idea of approximating the integrand with a polynomial. Using a sth order polynomial results in a s+1th order method. There are two types of Adams methods, explicit and implicit. The explicit are called AB and the implicit are called Adams-Moultoun (AM) methods. We will focus on the AB methods here.

We will not derive the AB methods as this is too time-consuming and is unnecessary for this paper. However if you wish to learn more about AB methods I highly recommend spending some time learning how to derive them.

The AB methods are explicit methods. The coefficients are $a_{s-1} = -1$ and $a_{s-2} = ... = a_0 = 0$, while b_j are chosen such that the methods have order s, which determines the methods.

The AB methods with s=1,2,3,4 are

$$\begin{aligned}
y_{n+1} &= y_n + hf(t_n, y_n) \\
y_{n+2} &= y_{n+1} + hf(\frac{3}{2}f(t_{n+1}, y_{n+1}) - \frac{1}{2}f(t_n, y_n)) \\
y_{n+3} &= y_{n+2} + hf(\frac{23}{12}f(t_{n+2}, y_{n+2}) - \frac{4}{3}f(t_{n+1}, y_{n+1}) \\
&\quad + \frac{5}{12}f(t_n, y_n)) \\
y_{n+4} &= y_{n+3} + hf(\frac{55}{24}f(t_{n+3}, y_{n+3}) - \frac{59}{24}f(t_{n+2}, y_{n+2}) \\
&\quad + \frac{37}{24}f(t_{n+1}, y_{n+1}) - \frac{3}{8}f(t_n, y_n))
\end{aligned}$$

We will now attempt to solve a problem using the AB4 method outlined above. Consider the IVP

$$y' = 7t^2 - \frac{4y}{t}, \quad 1 \geq t \geq 6, \quad y(1) = 5p$$

We first find the exact solution to this problem

$$\begin{aligned}
&y' + \frac{4y}{t} = 7t^2 \\
&\mu(t) = e^{\int \frac{4}{t}dt} = t^4 \\
&(t^4 y)' = 7t^6 \\
&t^4 y = t^7 + c \rightarrow y = t^3 + \frac{c}{t^4} \\
&y(1) = 5p = 0.62 \\
&0.62 = 1 + c \rightarrow c = -.38 \\
&y = t^3 - \frac{19}{50t^4}
\end{aligned}$$

Now we will apply our AB4 method

$$\begin{aligned}
y_{n+4} &= y_{n+3} + hf(\frac{55}{24}f(t_{n+3}, y_{n+3}) - \frac{59}{24}f(t_{n+2}, y_{n+2}) \\
&\quad + \frac{37}{24}f(t_{n+1}, y_{n+1}) - \frac{3}{8}f(t_n, y_n))
\end{aligned}$$

We apply it for $N = 16, 32, 64, 128, 256$ and calculate the rate of convergence as we did above

$$\tilde{p}_n = \frac{1}{ln(2)} ln\Big(\frac{||e_{N/2}||_\infty}{||e_N||_\infty}\Big)$$

Table III
EULER'S METHOD

N-Values	MaxError	Convergence Rate
16	1.26E-01	
32	1.09E-02	3.52
64	1.07E-03	3.35
128	8.71E-05	3.62
256	6.57E-06	1.73

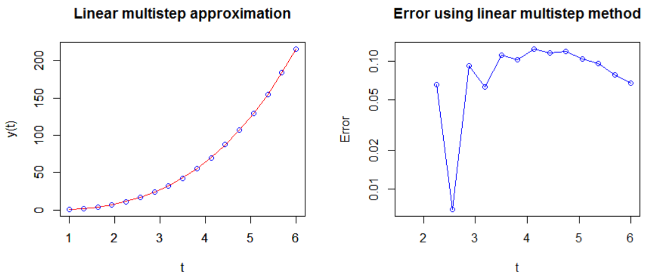

Figure 6. At N=16 we already have quite an accurate representation

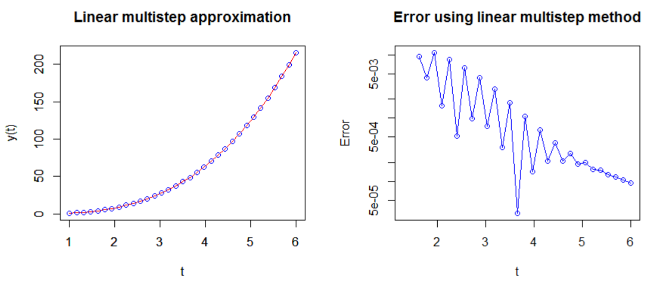

Figure 7. Here at N=32 we have some oscillation in our error but it is decreasing with larger t

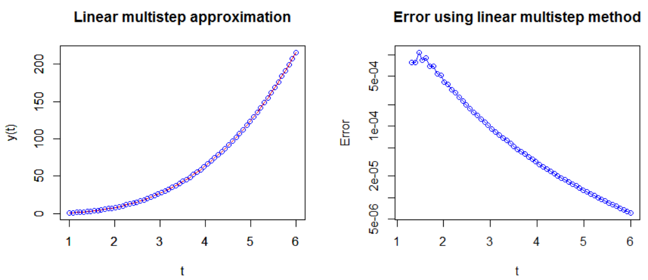

Figure 8. At N=64 we have a steady exponential decline in error

We will now have a look when our initial condition is $y(1) = 1$

$$
\begin{aligned}
& y' + \frac{4y}{t} = 7t^2 \\
& \mu(t) = e^{\int \frac{4}{t} dt} = t^4 \\
& (t^4 y)' = 7t^6 \\
& t^4 y = t^7 + c \to y = t^3 + \frac{c}{t^4} \\
& y(1) = 1 \\
& 1 = 1 + c \to c = 0 \\
& y = t^3
\end{aligned}
$$

which we approximate exactly due to the degree of t.

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